09/09/20

[JL70], [Bum98, 4.4], [PS83]

- Our goal classify irred. adm. rep'n of GL_2F. F narc. local field.
  [PS83] work with F finite field.

Not'n.

- $G_F = GL_2(F)$
- $D_F = \{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \}$
- $B_F = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \}$
- $N_F = \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \}$
  $F \simeq N_F \quad x \mapsto n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

Not'n. • let $X \in \mathrm{Top}$, $V \in \mathrm{Vect}_{\mathbb{C}}$.

$\mathrm{Map}(X,V)$ the set of V-val. fns.

$C^\infty(X,V)$ $\mathrm{Map}^\infty(X,V)$: smooth = loc. con.

$S(X,V)$ $\mathrm{Map}_c^\infty(X,V)$: smooth + cptly supported.

Strategy: [PS83, 13]

1) Use general methods, fr repns of D_F.
2) $D_F \rightsquigarrow B_F$. $\mathrm{Ind}_{B_F}^{G_F} \rho \leftrightarrow$ Find irr. rep. of G_F in
3) find at repns not constructed via this method.
   c. supercuspidal rep'ns.

Today: • $\{$Whittaker models$\} \leftrightarrow \{\mathrm{Ind}_{N_F}^{G_F} \psi\}$

- Uniquess of Whittaker & Kirillov.
- Comment on their relation

$\Big\}$ [S, God]

I. Recoll'n of Kirillov models

<u>Def'n</u>: A Kirillov model is rep'n of G_F on subspace of $\mathrm{Map}(F^\times, \mathbb{C})$ s.t. its restr'n $P_F \hookrightarrow G_F$ is iso to.

$$(\xi_\psi, \mathrm{Map}(F^\times, \mathbb{C}))$$

[JL, 2.9.1] Rep'n $(\xi_\psi, \mathrm{Map}_c^\infty(F^\times, \mathbb{C}))$ is irreducible the action is given by

$$\left(\xi_\psi\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} f\right)(v) = \underbrace{\psi}_{\text{fixed add. char of } F}(vx)\, f(va)$$

<u>Thm</u> (π, V) is irr. inf. adm. rep'n. of G_F.
Then π has a unique Kirillov model.

<u>Pf sketch</u>:
- V' as a subspc. of $\mathrm{Map}(F^\times, X)$
- X is 1.d.
- $X = V/V_0$

<u>What is X?</u>
- Can define (twisted) Jacquet functor.
  - i) ψ is add. char on F
  - ii) Can regard $\mathbb{C}$ as a N_F-module $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \psi_0(x)$

<u>Def'n</u>: J_ψ: $\mathrm{Mod}_{N_F} \underset{\leftarrow}{\overset{\perp}{\rightarrow}} \mathrm{Mod}_{\mathbb{C}}$ $\quad V \mapsto V \otimes_{N_F} \mathbb{C} =: J_\psi V$ jacquet module.

- $\eta: V \to V \otimes_{N_F} \mathbb{C}$. in $\mathrm{Mod}_{\mathbb{C}} = \mathrm{Vect}_{\mathbb{C}}$.

Can describe the kernel of η by [Bum 4.4.1]

• (π, V) irr. inf. adm. G_F.

I. Whittaker funtials.

Def'n A whitt. f'nl on V, s.t.

$$L\left(\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\cdot v\right) = \psi(x)\, Lv.$$

N_F

Def'n Define a subspace of $\mathrm{Map}(G_F, \mathbb{C})$

$$\mathcal{W}(\psi) := \left\{ W \;\middle|\; W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\cdot g\right) = \psi(x) W(g) \right\}$$

• $\mathcal{W}(\psi)$ regard as G_F module by right regular action ρ.
$\left(\rho(h)\, W(g) = W(gh)\right)$

Def'n A whittaker model of π is a submodule of $\mathcal{W}(\psi)$ iso. to (π, V).

• (π, V) these are all true
  - $J_{N,\psi} V$ is 1d.
  - $\Updownarrow$ V has a unique Kirillov model $\subseteq \mathrm{Map}(F^\times, \mathbb{C})$
  - $\Updownarrow$ V has " whittaker model. $\subseteq \mathrm{Map}(G_F, \mathbb{C})$.

• I'll assume Kirillov model $\exists$.

• Whitt f'nl dim 1 $\Rightarrow$ uniqueness of each of these model.

Def: A whitt. f'nl on V, s.t.
$L(\pi(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix})\cdot v) = \psi(x) Lv$.
N_F

II Properties of Whitt. fnl.

Prop'n: 1) The space of Whitt is 1d.

2) Let (π, V) be in Kirillov form.
The Whitt'n are precisely of the form,
$$L\phi = \lambda \phi(1)$$

Pf. Step 1 Setup.
- V in Kirillov form $V \subseteq Map(F^\times, \mathbb{C})$
- L be our whittaker f'nl. WTS: $L\phi = \lambda\phi(1)$.

Step 2. Use twist + Schwartz Space $:= Map_c^\infty(F^\times, \mathbb{C}) \subseteq V$.

Step 2a: recall'n [2.13.3, JL] $n_x := (\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix})$
- L is a lin. fnl on $Map_c^\infty(F^\times, \mathbb{C})$, $L(n_x \cdot \phi) = \psi(x) L\phi$,

then $L\phi = \lambda\phi(1)$.

Step 2b twist: $\forall \phi \in V \subseteq Map(F^\times, \mathbb{C})$

1) $\phi(a) = 0 \quad \forall |a| \gg 0$. $\quad |\cdot|$ is the abs. value on loc. field F.

2) $g \in N_F \quad \phi - g\cdot\phi \in Map_c^\infty(F^\times, \mathbb{C})$.
$*$

Step 2c: Let $\phi \in V$, x s.t. $\psi x \neq 1$. (I'll omit the π, $g\phi$, $\pi g \cdot \phi$)

$$L\phi = L(\phi - n_x\phi) + L(n_x\phi)$$
$*)$ + lemma 2a $\quad$ whitt.
$$= \lambda(\phi - n_x\phi)(1) + \psi x\, L\phi$$

$\phi = (\phi - n_x\phi) + n_x\phi$.
$\quad \cap$
$Map_c^\infty(F^\times, \mathbb{C})$

Hence, $(1-\psi x)L\phi = \lambda(1-\psi(x))\phi(1)$.

Since $\psi x \neq 1 \Rightarrow L\phi = \lambda\phi(1)$.

Make this precise next time.

$$\left\{\begin{matrix}\text{Kirillov}\\\text{models}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Whitt' fnls}\\ V\end{matrix}\right\} \longleftrightarrow \{\text{Whitt models}\}.$$

$$L_\psi \mapsto \{W_v : W_g = L_\psi(gv), v \in V\}$$

$$\subseteq \mathrm{Map}(G_F, \mathbb{C}).$$

$$V \longmapsto \begin{matrix}\phi \mapsto \phi(1)\\ \phi \in V.\end{matrix}$$

Kir Mod. $\longrightarrow$ Whit Model

$V' \subseteq \mathrm{Map}(F^\times, \mathbb{C})$ $\qquad\qquad$ $W' \subseteq \mathrm{Map}(G_F, \mathbb{C})$

$G_F \circlearrowright$

$\uparrow$

Ω_F

$$\phi \longmapsto W_\phi : g \mapsto (g \cdot \phi)(1).$$

$$N_F = \left\{\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\right\}.$$