Siegel modular forms

$$SL_2(\mathbb{R}) \circlearrowright \mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\} \cong SL_2(\mathbb{R})/O(2)$$

$$\gamma \in SL_2(\mathbb{R}), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \qquad \gamma.\tau = \frac{a\tau + b}{c\tau + d}$$

$$SL_2(\mathbb{R}) \supset SL_2(\mathbb{Z}) = \Gamma_1$$

$$f(\gamma.\tau) = (c\tau + d)^k f(\tau) \qquad \forall \gamma \in SL_2(\mathbb{Z}).$$

+ hol. + hol. at cusp. (full level)

f is an elliptic modular form of weight k and level Γ_1

Generalise $Sp_{2g}(\mathbb{R})$ g called the genus or degree.

isotropy group of the symplectic space $\mathbb{R}^{2g}$

basis $e_1, \cdots, e_g, f_1, \cdots, f_g$ s.t.

$$\langle e_i, e_j \rangle = 0$$
$$\langle f_i, f_j \rangle = 0$$
$$\langle e_i, f_j \rangle = \delta_{ij}$$
$$\langle f_j, e_i \rangle = -\delta_{ij}.$$

$Sp_{2g}(\mathbb{Z})$ preserves $\mathbb{Z}^{2g}$

$Sp_{2g}(\mathbb{R}) \ni \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$ $\quad a, b, c, d \in Mat_g(\mathbb{R})$.

$${}^t\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & I \\ -I & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix}$$

$$\overset{\parallel}{\begin{pmatrix} -{}^tca + {}^tac & -{}^tcb + {}^tad \\ -{}^tda + {}^tbc & -{}^tdb + {}^tbd \end{pmatrix}}$$

$$\Rightarrow \begin{cases} {}^tac \text{ sym}, \; {}^tbd \text{ sym} \\ {}^tad - {}^tcb = I_g \end{cases}$$

Siegel upper half plane. space.

$Sp_{2g}(\mathbb{R}) \circlearrowright \mathcal{H}_g = \{ \tau \in Mat_g(\mathbb{C}) \mid {}^t\tau = \tau, \ \operatorname{Im}\tau > 0 \} \cong Sp_{2g}(\mathbb{R}) / U(g)$

positive definite

$\mathcal{H}_1 = \mathcal{H}$. $\quad g=1 \quad SL_2(\mathbb{R})/U(1)$

$U(1) = SO(2)$.

$\gamma \in Sp_{2g}(\mathbb{R}) \qquad \gamma.\tau = (a\tau + b)(c\tau + d)^{-1}$.

$\overset{\parallel}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$

This is an action of $Sp_{2g}(\mathbb{R}) \circlearrowright \mathcal{H}_g$.

$\subset GL_g(\mathbb{C}) \ni a + ib$

Stabiliser of iI_g $= U(g) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in Sp_{2g}(\mathbb{R}) \;\middle|\; {}^taa + {}^tbb = I_g \right\}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . iI_g = iI_g \qquad (ia + b)(ic + d)^{-1} = iI_g \Rightarrow \begin{cases} a = d \\ b = -c \end{cases}$

$ia + b = -c + id$.

Siegel modular forms

?? how to generalise this

$$f(\gamma.\tau) = \boxed{(c\tau+d)^k} f(\tau) \quad \forall \gamma \in SL_2(\mathbb{Z}) \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

ell. mod. forms of weight k and level $\Gamma_1 = Sp_2(\mathbb{Z})$.

$\quad\quad$ $\|$ irred. repn. of $GL_1(\mathbb{C})$.

Siegel modular forms of weight ρ and level $\Gamma_g = Sp_{2g}(\mathbb{Z})$

$\quad\quad$ $\|$ irred. repn. of $GL_g(\mathbb{C})$ ← complexification of $U(g)$

$$f : \mathcal{H}_g \longrightarrow V_\rho$$

$$f(\gamma.\tau) = \rho(c\tau+d) f(\tau). \quad \forall \gamma \in \Gamma_g \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$+$ hol. $\quad$ no need for the "hol at cusp" condition b/c it's automatic when $g \geq 2$.

when $\rho = \det^k$, we get a (scalar-valued) Siegel modular form of weight k.

: $M_\rho(\Gamma_g)$,

$\bigoplus_\rho M_\rho(\Gamma_g)$ no ring structure. $\rho_1 \otimes \rho_2$ may be reducible.

Fourier expansion

$f \in M_\rho(\Gamma_g)$. $\quad b \in$ sym $g\times g$, integral matrix.

$$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in \Gamma_g.$$

$$f\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\cdot\tau\right) = f(\tau).$$

$$\overset{\|}{f(\tau+b)}$$

$$\Rightarrow \quad f(\tau) = \sum_{n \text{ half integral}} a(n)\, e^{2\pi i \operatorname{Tr}(n\tau)}$$

n sym. $g\times g$, $\quad n_{ij} \in \frac{1}{2}\mathbb{Z} \quad i \neq j. \quad n_{ii} \in \mathbb{Z}.$

$$\operatorname{Tr}(n\tau) = \sum_{i,j} n_{ij}\tau_{ji} = \sum_i n_{ii}\tau_{ii} + 2\sum_{i<j} n_{ij}\tau_{ij}.$$

We get all linear forms on $\mathcal{H}_g$ w/ integer coefficients for τ_{ij}. $\forall i,j$.

$$V_\rho \ni a(n) = \int_{\substack{x \bmod 1 \\ -\frac{1}{2}\leq x_{ij}\leq\frac{1}{2}}} f(\tau)\, e^{-2\pi i \operatorname{Tr}(n\tau)}\, dx$$

$u \in GL_g(\mathbb{Z})$.

$$a({}^t u n u) = \int f(\tau) e^{-2\pi i \operatorname{Tr}({}^t u n u \tau)} dx$$
$$= \int f(\tau) e^{-2\pi i \operatorname{Tr}(n \boxed{u \tau {}^t u})} dx.$$
$$= \int f\left(\begin{pmatrix} u^{-1} & \\ & {}^t u \end{pmatrix}\begin{pmatrix} u & \\ & {}^t u^{-1} \end{pmatrix}.\tau\right) e^{-2\pi i \operatorname{Tr}(n u \tau {}^t u)} dx$$
$$= \int \rho({}^t u) f(\tau) e^{-2\pi i \operatorname{Tr}(n\tau)} dx$$
$$= \rho({}^t u)\, a(n)$$

$$\begin{pmatrix} u & \\ & {}^t u^{-1} \end{pmatrix}.\tau = u \tau {}^t u.$$

Take $u = -I$.

<u>Cor</u>: $M_k(\Gamma_g) = 0$. if kg odd.

$g=1$ we require in the defn. of ell. mod. forms : hol at ∞.

$\Rightarrow a(n) = 0$ if $n \ngeq 0$

$g>1$ No cond. at cusp. $\Rightarrow$ Koecher's principle
$a(n) = 0$ if $n \ngeq 0$.
(n is not pos. semi-definite.)

$g=1$ $M_k(\Gamma_1) = 0$ if $k<0$.

$g>1$ $M_k(\Gamma_g) = 0$ if $k<0$.

ρ nontriv. $M_\rho(\Gamma_g) = 0$ if $\lambda_g \leq 0$.

$\rho \longleftrightarrow \lambda_1 \geq \cdots \geq \lambda_g$ highest weight.