Eisenstein series

How to construct a modular form of wt k.

Need $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$

For $g \in SL_2(\mathbb{R})$, define $f|_k g\,(z) := (c_g z + d_g)^{-k} f(g.z)$

$g.z = \dfrac{a_g z + b_g}{c_g z + d_g} \qquad g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}.$

This is a gp action of $SL_2(\mathbb{R})$ on hol. functions on $\mathfrak{h}$ w/ subexp. growth.

$f|_k g_1|_k g_2 = f|_k g_1 g_2$

$f_k|\gamma = f \qquad \forall \gamma \in \Gamma_1.$

Modular forms $=$ Γ_1-invariant functions in $\boxed{}$

Idea: take ϕ in $\boxed{}$ and then average over Γ_1.

$$f_\phi(z) := \sum_{\gamma \in \mathrm{Stab}_{\Gamma_1}\phi \backslash \Gamma_1} \phi|_k \gamma\,(z)$$

when ϕ is a rational function, this is called a <u>Poincaré series</u>.

Take $\phi = 1 \rightsquigarrow$ Eisenstein series of wt k E_k.

$$1|_k \gamma = (c_\gamma z + d_\gamma)^{-k}$$

$$E_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1} (c_\gamma z + d_\gamma)^{-k} \qquad \Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}) \right\}$$

$$\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + cn & b + dn \\ c & d \end{pmatrix}$$

$n \in \mathbb{Z}$

$$\left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}_{n \in \mathbb{Z}} \right\} \backslash \Gamma_1 \longleftrightarrow \{ (c, d) \in \mathbb{Z}^2, \ c, d \text{ coprime} \}$$

$$E_k(z) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ c, d \text{ coprime}}} \frac{1}{(cz + d)^k}$$

This is a modular form of wt k.

abs. conv. when $k \geq 4$

cond. conv. when $k = 2$.

$$G_k(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k}$$

$$G_k(z) = \frac{1}{2} \sum_{r=1}^{\infty} \sum_{\substack{c,d \in \mathbb{Z} \\ c,d \text{ coprime}}} \frac{1}{r^k (cz+d)^k}$$

$$= \sum_{r=1}^{\infty} \frac{1}{r^k} \cdot E_k(z) = \underbrace{\zeta(k)}_{\text{Riemann zeta function.}} E_k(z).$$

$$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z)$$