Global Whittaker models

π adm. $\mathcal{H}(\mathbb{A})$-repn. w/ central char ω.

A space of functions on $G(\mathbb{A})$, K-fin. sm, even analytic at arch place

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \tau(x) W(g)$$

$$W\left(\begin{pmatrix} x & \\ & x \end{pmatrix} g\right) = \omega(x) W(g)$$

$$W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} g\right) = O(|x|^{-N}) \quad \forall N \quad |x| \to \infty.$$

Realise π in the space of Whittaker functions.

$\pi \simeq \bigotimes_v' \pi_v$ (Flath) a.e. v π_v unramified.

Assume π_v has a Whittaker model. $W(\pi_v)$.

π has a global Whittaker model $\bigotimes_v' W(\pi_v)$

w.r.t. W_v^0. v π_v unramified τ_v unramified.

$$W_v^0\left(\underbrace{\begin{pmatrix} a & \\ & a \end{pmatrix}}_{\omega_v}\begin{pmatrix} b & \\ & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}}_{\tau_v(c)} \underbrace{k}_{\text{inv.}}\right)$$

Casselman–Shalika formula.
for GL_2-case see Sec 16.
$W_v^0(1) = 1$.

Global Whittaker model is unique.

Have one already $\mathcal{W} := \bigotimes_v' \mathcal{W}(\pi_v)$ $\qquad \mathcal{W}_v := \mathcal{W}(\pi_v)$

Assume we have another one $\mathcal{W}'$.

$$\mathcal{W} \xrightarrow{\iota \atop \sim} \mathcal{W}' \qquad (\mathcal{H}(\mathbb{A})\text{- equivariant})$$

Fix v place. For $x \neq v$, fix wh. fun. W_x.

$$\mathcal{W}_v \longrightarrow \mathcal{W} \xrightarrow{\iota \atop \sim} \mathcal{W}' \longrightarrow \mathcal{W}_v'$$

$$\left(g_v \mapsto W_v(g_v)\right) \mapsto \left(g \mapsto W_v(g_v) \prod_{x \neq v} W_x(g_x)\right) \mapsto W' \mapsto g_v \mapsto W'(g_v \prod_{x \neq v} g_x)$$

local $\Rightarrow$ uniqueness

$$\iota\Big(W_v \cdot \prod_{x \neq v} W_x\Big)(g) = \overbrace{c(g, v, W_{x \neq v})}^{\text{scalar}} \cdot W_v(g_v)$$

For any fixed g_x $x \neq v$.

does not depend on W_v.

S = fin. set of places

$$W_S \in \bigotimes_{v \in S} \mathcal{W}_v \qquad \iota(W_S \cdot W^S)(g) = c(g, S, W^S)\, W_S(g_S)$$

Fix $W^S = \bigotimes_{x \notin S}' W_x$ $\qquad \iota(W_S \cdot W_{v'} \cdot W^{S'})(g) = c(g, S \cup \{v'\}, W^{S'}) W_S(g_S) W_{v'}(g_{v'})$

$S' = S \cup \{v'\}$

$\Rightarrow \quad c(g, S, W^S) = c(g, S \cup \{v'\}, W^{S'})\, W_{v'}^{\circ}(g_{v'})$

$\Rightarrow \quad c(g, S, W^S) = \underbrace{c \prod_{x \notin S} W_x^{\circ}(g_x)}$ dep. only on ι.

$\Rightarrow \quad \iota(W)(g) = c\, W(g) \quad \Rightarrow \quad W' = W$. get global uniqueness. (true for other reductive groups)

Ref: Shalika Multiplicity One Theorem for $GL(n)$

Multiplicity One Theorem

Thm 3: The multiplicity of irreducible subrepn. of $G(\mathbb{A})$ in $L_0^2(G(k)\backslash G(\mathbb{A}), \omega)$ is one. (not true in general for other reductive groups)

Pf: $\underset{\text{irred.}}{V} \subseteq L_0^2 \rightsquigarrow$ pass to K-fin. vectors $\mathcal{H}(\mathbb{A}) \underset{\pi}{\circlearrowright} V^{fin} \subseteq V$.

cuspidal functions in $L_0^{2,fin}$ are smooth, analytic at arch places rapidly decreasing. (these are called cuspidal forms)

$$\varphi \in V^{fin}.$$

$$\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \sum_{\xi \neq 0} W_\varphi\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}\right) \tau(\xi x)$$

Recall:

$$W_\varphi(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \overline{\tau(x)}\, dx$$

This is a Whittaker function. From properties of φ

$\Rightarrow W_\varphi$ K-fin, sm. analytic at arch. places w/ central char. rapidly dec.

$\pi \longrightarrow W(\pi)$ get global Whittaker model for π.

$\varphi \longmapsto W_\varphi$

Uniqueness of $W(\pi)$

$$\varphi(g) = \sum_{\xi \neq 0} W_\varphi\left(\begin{pmatrix} \xi & \\ & 1 \end{pmatrix} g\right)$$

π has a unique realisation in $L_0^{2, fin}$.

$\square$