$L^2_0(G(k)\backslash G(\mathbb{A}), \omega)$ not nice

Want repn on <u>smooth</u> functions on $G(\mathbb{A})$

$G(\mathbb{A}_\infty) \times G(\mathbb{A}_f)$

smooth loc. constant

<u>Admissible Repns</u>

non-arch $G := GL_2(k_v)$

$G \overset{\pi}{\curvearrowright} V$ complex v.s.

① $\forall v \in V$ $\exists K'$ open cpt subgp that fixes v

② $\forall K'$ open cpt subgp $V^{K'}$ fin. dim.

① ② $\iff$ $V = \bigoplus_{\vartheta \in \hat{K}} V(\vartheta)$ $V(\vartheta)$ fin dim.

$G \curvearrowright V \rightsquigarrow \mathcal{H} := C_c^\infty(G) \overset{\pi}{\curvearrowright} V$ $\bigcup_{K'}$ bi-K'-inv. fun.

① $\forall v \in V$ $\exists f \in \mathcal{H}$ s.t. $\pi(f)v = v$.

② $\forall f \in \mathcal{H}$ $\pi(f)V$ fin. dim.

$\mathcal{H} \overset{\pi}{\circlearrowright} V$ w/ ① $\rightsquigarrow$ $G \overset{\pi}{\circlearrowright} V$ w/ ①

Desired : $\pi(g)$

$$\pi(f)\, v = \int_G f(g')\pi(g')\, v\, dg'$$

$$\pi(g)\pi(f) v = \int_G f(g')\, \pi(g)\pi(g')\, v\, dg'$$

$$= \int_G f(g')\, \pi(gg')\, v\, dg'$$

$$= \int_G \underbrace{\boxed{f(g^{-1}g')}}_{\lambda(g) f(g')} \pi(g')\, v\, dg'$$

$$= \pi(\lambda(g) f)\, v$$

Defn : $G \circlearrowright V$ adm if ① ② hold.

$\mathcal{H} \circlearrowright V$ adm if ① ② hold.

$\pi(g)\, v := \pi(\lambda(g) f)\, v$ (choose f s.t. $\pi(f) v = v$)

Well-defined

$\mathcal{H} \circlearrowright V$ w/ ① ② $\rightsquigarrow$ $G \circlearrowright V$ w/ ① ②

arch $\quad G \underset{\text{cont.}}{\curvearrowright} V$ (complete, separable, loc. convex TVS)

$\mathcal{H}^{JL}$ gen. by $\mathcal{H}_1, \mathcal{H}_2$ $\left[\begin{array}{l} \mathcal{H}_1 := C_c^\infty(G)^{\text{bi-}K\text{-fin}} \\ \mathcal{H}_2 := \{\text{matrix coeff. of } K\} \underset{\text{dense}}{\subseteq} C(K) \end{array}\right.$

$G \curvearrowright V \quad \rightsquigarrow \quad \mathcal{H}^{JL} \curvearrowright V \supseteq V^{K\text{-fin}} \circlearrowleft \mathcal{H}^{JL}$

$V^{K\text{-fin}} = \{ v \in V \mid \pi(K) v \text{ fin dim} \}$

$\mathcal{H}^{JL} \curvearrowright V$ complex v.s.

① $\forall v \in V \quad \exists f_i \in \mathcal{H}_1$ s.t. $\sum_{i=1}^{r} \pi(f_i) v = v$

② $\forall \xi$ elem. idempotent $\quad \pi(\xi) V$ fin. dim

③ $\forall \xi$ elem. idempotent $\quad v \in V \quad \xi \mathcal{H}_1 \xi \xrightarrow{\text{cont.}} \pi(\xi) V$, $\quad f \mapsto \pi(f) v$

$\forall \vartheta \in \hat{K}, \quad \chi_\vartheta(k) := \dim(\vartheta) \operatorname{Tr} \vartheta(k)$

If ξ = fin. sum of χ_ϑ's, then ξ is said to be an elem. idem.

$\pi(\chi_\vartheta) V = V(\vartheta)$

<u>Defn</u>: $\mathcal{H}^{JL} \curvearrowright V$ s.t. ① ② ③ hold is called <u>admissible</u>.

$\mathcal{H}^{JL} \circlearrowright V$ adm. $\rightsquigarrow$ $U(\mathfrak{g}) \circlearrowright V$ $(\mathfrak{g}, K)$-mod.
K, Z $\mathfrak{g} = \mathrm{Lie}\, G$

$\pi(X)v := \pi(X * f)v$ (choose $f \in \mathcal{H}$, s.t. $\pi(f)v = v$)
well-defined

δ_g Dirac delta distribution at $g \in G$.

$\delta_g * f = \lambda(g) f$.

$\pi(\delta_g)v := \pi(\boxed{\delta_g * f})v$ $(\pi(f)v = v)$
may not be bi-K-fin. if $g \in G$.
$g \in K, Z$ O.K.

$\mathcal{H}^{JL} \circlearrowright V$ adm $\rightsquigarrow$ $(\mathfrak{g}, K)$-module adm. $\left(\begin{array}{l} V = \bigoplus_{\vartheta \in \hat{K}} V(\vartheta) \\ V(\vartheta) \text{ fin dim} \end{array} \right)$

Given $(\mathfrak{g}, K)$-mod. adm. $\exists^{\text{(canon)}}\, G \circlearrowright V^{CW}$ (a sm. Fréchet space of moderate growth)

s.t. $G \circlearrowright V^{CW} \rightsquigarrow (\mathfrak{g}, K)$-mod $V^{CW, K\text{-fin}}$

(Thm of Casselman - Wallach)

Whittaker models

π_v $\quad \mathcal{H}_v \circlearrowright V_v$ adm. central char $= \omega_v$

Space of Whittaker fun = Whittaker fun w/ central char ω_v

$$\left\{ \begin{array}{l} \text{fun. } W_v \text{ on } G_v \text{ s.t. right-}K_v\text{-fin. , } C^\infty \text{ (even analytic) if } v \text{ arch.} \\ W_v\left(\begin{pmatrix} 1 & x_v \\ & 1 \end{pmatrix} g_v\right) = \tau_v(x_v) W_v(g_v) \\ \text{growth cond. } W_v\left(\begin{pmatrix} x_v & \\ & 1 \end{pmatrix}\right) = \begin{cases} 0 & \text{for } |x_v| \text{ large } v \text{ non-arch} \\ O(|x|_v^{-N}) & \forall N \ |x_v| \text{ large } v \text{ arch} \end{cases} \\ \text{central char } W_v\left(\begin{pmatrix} x_v & \\ & x_v \end{pmatrix} g_v\right) = \omega_v(x_v) W_v(g_v) \end{array} \right\}$$

$O(|x|_v^{-N})$ ← rapidly decreasing

$\pi_v \simeq \underbrace{W(\pi_v)}_{\text{Whittaker model of } \pi_v}$ realised in the space of Whittaker functions.

Local (last semester) Whittaker model of π_v is unique. (if it exists) mult ≤ 1.