L-series

$M_k(\Gamma_1)$ is spanned by normalised Hecke eigenforms

$$f = \sum_{n=0}^{\infty} a_n q^n \quad \text{satisfying}$$

$$a_m a_n = \sum_{r \mid \gcd(m,n)} r^{k-1} a_{mn/r^2} \qquad \text{for all } m,n \geq 1$$

Special cases:

① if $\gcd(m,n)=1$ then $a_m a_n = a_{mn}$ (multiplicative)

② if $m=p$, $n=p^\nu$: $a_p a_{p^\nu} = \boxed{a_{p^{\nu+1}}} + p^{k-1} a_{p^{\nu-1}}$ $\forall$ primes p $\forall \nu \geq 1$

Consider Dirichlet series $\boxed{L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}}$ $s \in \mathbb{C}$

① $\Rightarrow$ $$L(f,s) = \prod_{p \text{ prime}} \boxed{\left(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \dots\right)}$$ Euler product

② $\Rightarrow$ $$\sum_{\nu=0}^{\infty} a_{p^\nu} x^\nu = \frac{1}{1 - a_p x + p^{k-1} x^2}$$

① + ② $\Rightarrow$ $$\boxed{L(f,s) = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}}$$ Euler product

<u>ex</u>: $f = G_k$ Eisenstein series $\qquad a_1(G_k) = 1$

$$L(G_k, s) = \prod_p \frac{1}{1 - (p^{k-1}+1) p^{-s} + p^{k-1-2s}} = \prod_p \frac{1}{(1-p^{-s})(1-p^{k-1-s})}$$

$$= \left(\prod_p \frac{1}{1-p^{-s}}\right)\left(\prod_p \frac{1}{1-p^{k-1-s}}\right) = \zeta(s)\,\zeta(s+1-k)$$

For what $s \in \mathbb{C}$ do the above actually make sense?

Let $f \in S_k(\Gamma_1)$. (not necessarily eigenform)

Theorem (Hecke) $|a_n| \leq \boxed{C n^{k/2}}$ $\forall n \geq 1$

(C depends on f).

Idea of proof: $z = x+iy \in \mathcal{H}$.

$z \longmapsto y^{k/2} |f(z)|$ is bounded on $\mathcal{H}$

$\Rightarrow \exists\, c > 0$ st. $|f(z)| \leq \boxed{c y^{-k/2}}$ $\forall z \in \mathcal{H}$.

Fourier coeffs:

$$a_n = e^{2\pi n y} \int_0^1 f(x+iy) e^{-2\pi i n x} dx$$

So $|a_n| \leq c y^{-k/2} e^{2\pi n y}$ $\forall n \geq 1, \forall y > 0$

Take $y = \frac{1}{n}$

$|a_n| \leq \underbrace{\boxed{c e^{2\pi}}}_{C} n^{k/2}$

<u>Corollary</u>: $L(f,s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ converges absolutely for $\mathrm{Re}(s)>1+\frac{k}{2}$.

Can we do better? (Spoilers: analytic continuation & functional equation)

Gamma function
$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}\,dt$$

$t\mapsto \lambda t$

$n\geq 1$,

For any $\lambda>0$,
$$\lambda^{-s}\Gamma(s)=\int_0^\infty t^{s-1}e^{-\lambda t}\,dt.$$
Take $\lambda=2\pi n$

$\boxed{\text{For } \mathrm{Re}(s)>1+\frac{k}{2}}$

$$(2\pi)^{-s}\Gamma(s)\,n^{-s}=\int_0^\infty t^{s-1}e^{-2\pi n t}\,dt$$

$t\mapsto\frac{1}{t}$

$$\underbrace{(2\pi)^{-s}\Gamma(s)L(f,s)}_{\boxed{L^*(f,s)}}=\sum_{n=1}^{\infty}a_n\int_0^\infty t^{s-1}e^{-2\pi n t}\,dt=\boxed{\int_0^\infty t^{s-1}f(it)\,dt}$$

converges abs. for all $s\in\mathbb{C}$

Substitution $t \mapsto \frac{1}{t}$ gives:

$$L^*(f, k-s) = (-1)^{k/2} L^*(f,s) \qquad \left(\begin{array}{c}\text{functional}\\ \text{equation}\end{array}\right)$$

$\exists$ converse theorem. $\qquad \Gamma_1 = SL_2(\mathbb{Z})$

Other L-series to know & love

Elliptic curve over $\mathbb{Q}$ $\qquad A, B \in \mathbb{Z}$,

$$E: \quad y^2 = x^3 + Ax + B \qquad \Delta = -4A^3 - 27B^2 \neq 0$$

For $p \nmid \Delta$, the reduction mod p

$$E_p: \quad y^2 = x^3 + \bar{A}x + \bar{B}$$

is an elliptic curve over $\mathbb{F}_p$

Set $a_p(E) = p - \#\{(x,y)\in(\mathbb{Z}/p\mathbb{Z})^2 \mid y^2 = x^3+\bar{A}x+\bar{B}\}$

$$L(E,s) = \prod_{p\nmid\Delta} \frac{1}{\boxed{1 - a_p(E)p^{-s} + p^{1-2s}}} \prod_{p\mid\Delta}\frac{1}{\boxed{\text{poly of deg} \le 2 \text{ in } p^{-s}}}$$

Taniyama: is there some relation $f \rightsquigarrow E/\mathbb{Q}$?

Galois reps

wt $k=2$

Theorem (Eichler–Shimura)

$f\in S_2(\Gamma_0(N))$ Hecke eigenform with $a_n\in\mathbb{Z}$ $\Longrightarrow$ $\exists\, E/\mathbb{Q}$ s.t. $L(E,s)=L(f,s)$.

$\Longleftarrow$?

Shimura-Taniyama-Weil conjecture

STW conjecture proved

by $\underbrace{\text{Wiles} \cup \text{Taylor-Wiles}}_{\sim 1996} \cup \underbrace{\text{Breuil-Conrad-Diamond-Taylor}}$

It was a major piece in the proof of Fermat's Last Theorem.