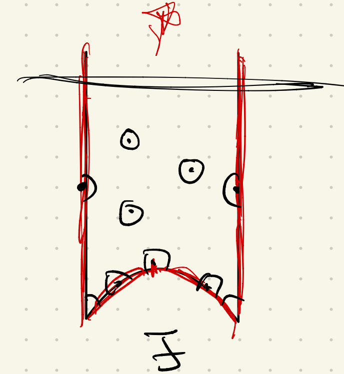
$\Gamma_1 = SL_2(\mathbb{Z})$

$\underline{\dim M_k(\Gamma_1) < \infty}$



issues:

- $\Gamma_1 \backslash \mathcal{H}$ not compact

  $$\overline{\mathcal{H}} = \mathcal{H} \cup \boxed{\mathbb{Q} \cup \{\infty\}}$$

  $$\Gamma_1 \backslash \overline{\mathcal{H}} = \overline{\Gamma_1 \backslash \mathcal{H}}$$

- $\Gamma_1 \backslash \mathcal{H}$ singular

$f : \mathcal{H} \to \mathbb{C}$

$f \in M_k(\Gamma_1)$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \forall z \in \mathcal{H} \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1$$

<u>value</u> of f at $P \in \Gamma_1 \backslash \mathcal{H}$ is not well-defined

<u>order of vanishing</u> of f at $P \in \Gamma_1 \backslash \mathcal{H}$ <u>is</u> well-defined

$$\underbrace{\mathrm{ord}_P(f)} := \underbrace{\mathrm{ord}_z(f)} \quad \text{for } P = \Gamma_1 z$$

Mantra: "the total number of zeros of f does not depend on f"

(only on Γ_1 and k).

$$\text{ord}_\infty(f) = \min\{n \mid a_n \neq 0\}$$

"$\displaystyle\sum_{P \in \overline{\Gamma_1 \backslash \mathcal{H}}} \text{ord}_P(f)$"

$$f(q) = \sum_{n=0}^{\infty} a_n q^n$$

Define $n_P = \#\,\text{Stab}(z) = \begin{cases} 1 & \text{else} \\ 2 & z = i \\ 3 & z = \omega \end{cases}$ for $P = \Gamma_1 z$

<u>Prop</u>: If $f \in M_k(\Gamma_1) \setminus \{0\}$, then

$$(*) \quad \sum_{P \in \Gamma_1 \backslash \mathcal{H}} \frac{1}{n_P} \text{ord}_P(f) + \text{ord}_\infty(f) = \frac{k}{12}$$

This is proved by contour integration.

Corollary: $\dim M_k(\Gamma_1) = 0$ if $k < 0$ or k is odd.

If $k \geq 0$ is even then

$$\dim M_k(\Gamma_1) \leq \left\lfloor \frac{k}{12} \right\rfloor + 1.$$

Proof: $\circledast$ $$\sum_{P \in \Gamma_1 \backslash \mathcal{H}} \frac{1}{n_P} \operatorname{ord}_P(f) + \operatorname{ord}_\infty(f) = \frac{k}{12}$$

$n_P = 1$

Write it as: $\frac{a}{3} + \frac{b}{2} + c = \frac{k}{12}$ $\quad a,b,c \in \mathbb{Z}_{\geq 0}$

(in fact $a, b \in \{0,1\}$).

$4a + 6b + 12c = k$ impossible if $k < 0$

if k is odd.

Set $m = \left\lfloor \frac{k}{12} \right\rfloor + 1$. Choose $P_1, \dots, P_m \in \Gamma_1 \backslash \mathcal{H}$ distinct s.t. $n_{P_i} = 1$.

Suppose $\dim M_k(\Gamma_1) > m$.

$\exists$ m+1 lin. indep. $\underbrace{f_1, \dots, f_{m+1}}$ $\in M_k(\Gamma_1)$.

$$\left\{ \begin{array}{l} a_1 f_1(P_1) + \dots + a_{m+1} f_{m+1}(P_1) = 0 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_1 f_1(P_m) + \dots + a_{m+1} f_{m+1}(P_m) = 0 \end{array} \right.$$

m equations in m+1 unknowns $a_1, \dots, a_{m+1}$

$\Rightarrow \exists (a_1, \dots, a_{m+1}) \neq (0, \dots, 0)$.

Set $f = a_1 f_1 + \dots + a_{m+1} f_{m+1}$, look at $\circledast$:

$$\underset{\geq m+1}{\text{LHS}} \neq \underset{\frac{k}{12} < m}{\text{RHS}}$$

, contradiction.

$$\frac{k}{12} = \frac{k}{4\pi} \underbrace{\mathrm{vol}(\Gamma_1 \backslash \mathcal{H})}_{= \pi/3} \qquad d\mu = \frac{dx\,dy}{y^2}$$

Take other Γ such that $\mathrm{vol}(\Gamma \backslash \mathcal{H}) < \infty$

Ex: $\Gamma \subset \Gamma_1 = SL_2(\mathbb{Z})$ of finite index.

Same approach gives $\dim(M_k(\Gamma)) < \infty$ for such Γ.