Intro: The 1-~~2~~-3 of modular forms

Congruences: some examples

For $n \geq 1$, $p(n) :=$ # partitions of n

Thm (Watson 1938): If $24n \equiv 1 \pmod{5^m}$ then

$$p(n) \equiv 0 \pmod{5^m}$$

1988

For $n \geq 1$, $c(n) := \dim$ (degree $(n+1)$ piece of the Monster vertex algebra)

$c(1) = 196884$

Thm (Lehmer 1949): If $n \equiv 0 \pmod{2^m}$ then

$$c(n) \equiv 0 \pmod{2^{3m+8}}.$$

$$\frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)\, q^n =: j(q)$$

j-invariant

(non-holom @ ∞) modular form of weight 0

$$\sum_{n=1}^{\infty} p\left(\frac{n+1}{24}\right) q^n = \frac{1}{\boxed{q^{1/24} \prod_{n=1}^{\infty} (1-q^n)}}$$

$p(\text{non-integer}) = 0$

$$\overset{!!}{=} \eta$$

Dedekind eta function, modular form of weight $\frac{1}{2}$

There are congruences between coeffs of modular forms

$$\Delta := \eta^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

Δ modular form of wt 12

$$\tau(n) \equiv n\,\sigma_9(n) \pmod{7}$$

$$\sigma_9(n) = \sum_{d \mid n} d^9$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

e.g. if n is prime

$$\sigma_9(p) = 1 + p^9$$

Serre & Swinnerton-Dyer 1972

Modular form of weight k:

$$\mathcal{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$$

$f : \mathcal{H} \ni z \longrightarrow \mathbb{C}$ analytic

$M_k \supset S_k$ cusp forms

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$

$z \in \mathcal{H}$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

$\boxed{f(z+1) = f(z)}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Fourier exp

$f(z) = \sum a_n e^{2\pi i n z}$

and f is "holomorphic @ ∞"

Eisenstein series: $E_k = \text{constant term} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ " $\sum a_n q^n$

wt k, $k \geq 4$

$$\Delta = \frac{E_4^3 - E_6^2}{12^3} \quad \text{weight } 12$$

$$q = e^{2\pi i z}$$

Harder: $\underline{f} = \dfrac{E_6 E_4^4 - E_6^3 E_4}{12^3} = q - 288 q^2 - 128844 q^3 + \ldots$ wt 22

$f \in S_{22}$

$S_k \circlearrowleft T_p$ for each prime p Hecke operators

$f \in \underbrace{S_{22}}_{\text{1-dim}}$ automatically an eigenvector

eigenvalue of T_p is the p^{th}-coefficient

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modular forms "live on" the space $SL_2(\mathbb{Z}) \backslash \mathcal{H}$ Riemann sphere without a point.

$\hookrightarrow$ sections of line bundles

$$\mathcal{H}_2 = \{ \underbrace{Z \in M_2(\mathbb{C})} \mid \underbrace{Z^t = Z}, \underbrace{\mathrm{Im}(Z) > 0}_{\text{positive definite}} \}$$

3-dimensional complex manifold.

$f : \mathcal{H}_2 \to \mathbb{C}$ analytic

Siegel modular forms of genus 2

$$f((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k f(Z)$$

$Z \in \mathcal{H}_2$ $\quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z})$

More generally, take a representation (τ, V) of $GL_2(\mathbb{C})$

$\tau = \det^k$

$f : \mathcal{H}_2 \to V$ analytic

$$f((AZ+B)(CZ+D)^{-1}) = \tau(CZ+D) f(Z)$$

magic occurs

Conjecture (Harder) Given $f \in S_{22}$, there exists a Siegel modular form F of weight $\tau = \mathrm{Sym}^4 \otimes \det^{10}$ such that

$$\lambda(p) \equiv a_p + p^{13} + p^{8} \pmod{41} \quad \text{for all primes } p$$

Bergström–Dummigan

Faber–van der Geer

$\lambda(p)$ = Hecke eigenvalue for F

a_p = Hecke eigenvalue for f

GL_2 $\quad GSp_4$

Chenevier–Lannes : proved Harder's conjecture