6 May 2021

L-functions I

(following Godement 6 & 7 in Chapter 3)

$G = GL_2$ / number field k

$K_v \subset GL_2(k_v)$ max compact, for each place v

Suppose we have:

- for each v, π_v irred admissible rep of the local Hecke algebra $\mathcal{H}_v$
- for almost all v, π_v contains the identity rep of K_v
- χ character of $\mathbb{A}^\times / k^\times$

  $\chi = \prod_v \chi_v$, χ_v char of $k_v^\times$, unramified for almost all v.

Set
$$L_\pi(\chi, s) = \prod_v \underbrace{L_{\pi_v}(\chi_v, s)}_{\text{local L-functions, def in Chapters 1 \& 2}}$$
formally!

Here $L_{\pi_v}(\chi_v, s)$ is given by: $\qquad s' = 2s - \frac{1}{2}$

$$
\begin{cases}
1 & v \text{ non-arch}, \pi_v \text{ supercuspidal} \\
L(\mu - \chi_v, s') & \text{"} \quad \pi_v = \pi_{\mu,\nu}, \ \mu\nu^{-1} = |x| \\
L(\nu - \chi_v, s') & \text{"} \quad \pi_v = \pi_{\mu,\nu}, \ \mu\nu^{-1} = |x|^{-1} \\
\boxed{L(\mu - \chi_v, s') L(\nu - \chi_v, s') \quad \text{"} \quad \pi_v = \pi_{\mu,\nu} \text{ principal series}} & \\
L(\mu - \chi_v, s') L(\nu - \chi_v, s') & v \text{ arch}, \ \pi_v = \pi_{\mu,\nu} \text{ principal series} \\
(2\pi)^{r - s' - t} \Gamma(s' + t - r) & v \text{ arch}, \ \pi_v = \sigma_{\mu,\nu} \text{ discrete series}
\end{cases}
$$

Let S = finite set of places $\supset$ {archimedean places}
and such that for all $v \notin S$:

- π_v contains the identity rep of K_v
- $\mathrm{Ker}(\tau_v) = \mathcal{O}_{k_v}$ $\qquad m_v = (\varpi)$
- χ_v is unramified $\qquad q = \#(\mathcal{O}_{k_v}/m_v)$

Then

$$
\begin{aligned}
L_{\pi_v}(\chi_v, s) &= L(\mu - \chi_v, s') L(\nu - \chi_v, s') \\
&= \frac{1}{1 - \frac{\mu}{\chi_v}(\varpi) q^{-s'}} \, \frac{1}{1 - \frac{\nu}{\chi_v}(\varpi) q^{-s'}}
\end{aligned}
$$

If π_v is a preunitary representation of $\mathcal{H}_v$,

$$|\mu(\varpi)| = q^{-\sigma_0/2}, \quad |\nu(\varpi)| = q^{\sigma_0/2}, \quad 0 \le \sigma_0 \le 1$$

$$L_\pi(\chi,s) = \underbrace{\prod_{v\in S}(\dots)}_{\text{finite product}} \boxed{\prod_{v\notin S}\left(1-\chi_v(\varpi)q^{-s'-\frac{\sigma_v}{2}}\right)^{-1}\left(1-\chi_v(\varpi)q^{-s'+\frac{\sigma_v}{2}}\right)^{-1}}$$

converges for $\mathrm{Re}(s)\gg 0$

$$s' = 2s-\frac{1}{2}$$

For $v\notin S$ we have
$$\varepsilon_{\pi_v}(\chi_v,s)=1.$$
So we get a finite product
$$\varepsilon_\pi(\chi,s) = \prod_v \varepsilon_{\pi_v}(\chi_v,s).$$

Recall: unitary rep of $G(\mathbb{A})$ on $L^2_0(G(k)\backslash G(\mathbb{A}),\omega)$

If π is an irreducible component, we denote by π_v the corresponding irred. admissible rep of $\mathcal{H}_v$.

This matches the above setup, so we get for any character χ of $\mathbb{A}^\times/k^\times$ an L-function
$$L_\pi(\chi,s) \quad \text{defined for } \mathrm{Re}(s)\gg 0.$$

Theorem 4: $L_\pi(\chi,s)$ is entire, bounded on every vertical strip, and satisfies the functional eq:
$$L_\pi(\chi,s) = \varepsilon_\pi(\chi,s)\,L_\pi(\omega-\chi,1-s)$$

Compare: for a Hecke L-function, $L(\chi,s)=\prod_v L(\chi_v,s)$
$$L(\chi,s)=\varepsilon(\chi,s)L(\chi^{-1},1-s).$$